In this section, we introduce the soft and collinear limits in the tree amplitude.

Soft and collinear limits are important concepts in modern quantum field theory. In these limits, a complicated often factorizes to simpler amplitudes. They have immediate usages:

- Soft and collinear limits can be used to check if an amplitude result is correct or not.
- From soft and collinear limit, often we can guess the expression of a complicated amplitude. This approach is called the amplitude "bootstrap".

Further, soft and collinear limits can be used to study the parton distribuction function in QCD. Nowadays, soft and collinear limits turn into an active research field called the soft and collinear effective theory (SCET).

I. SOFT LIMIT

We consider the soft limit of a tree gluon amplitude in Yang-Mills theory. The object under concern is the partial amplitude $A(1,\ldots,a,i,b\ldots n)$.

Consider the soft limit,

$$k_i^\mu \to 0\,. \tag{1}$$

Note that this is not a Lorentz invariant statement. The Lorentz invariant statement is

$$|s_{ij}| << |s_{kl}|, \quad j,k,l \neq i, \quad k \neq l \tag{2}$$

In this limit, $A(1,\ldots,a,i,b\ldots n)$ would be divergent. The reason for this divergence is that propagators like $1/(p_i+p_j)^2 \to \infty$.

The divergent propagator for this partial amplitude must be from the adjacent particles of p_i, i.e., p_a and p_b.

- The divergent term from $1/(p_i+p_a)^2$.

$$M(a,i)^\mu \frac{-i}{(p_i+p_a)^2} M(1,\ldots,p_a+p_i,b\ldots n)_\mu \tag{3}$$

  where $M(a,i)^\mu$ is the three-point vertex and $M(1,\ldots,p_a{+}p_i,b,\ldots n)_\mu$ is the sum of Feynman tree diagrams with gluons a,i combined.

  From color stripped Feynman rules,

$$M(a,i)^\mu = \frac{i}{\sqrt{2}}\Big((\epsilon_i\cdot\epsilon_a)(p_a-p_i)^\mu + \epsilon_i^\mu(2p_i-p_a)\cdot\epsilon_a + \epsilon_a^\mu(-2p_a-p_i)\cdot\epsilon_i\Big) \tag{4}$$

  Only the last term is non-vanishing and we consider it as the leading term for the soft limit.

- The divergent term from $1/(p_i+p_b)^2$.

$$M(i,b)^\mu \frac{-i}{(p_i+p_b)^2} M(1,\dots,a,p_i+p_b\dots n)_\mu \tag{5}$$

where $M(i,b)^\mu$ is the three-point vertex and $M(1,\dots,a,p_i+p_b,\dots n)_\mu$ is the sum of Feynman tree diagrams with gluons i,b combined.

From color stripped Feynman rules,

$$M(i,b)^\mu = \frac{i}{\sqrt{2}}\Big((\epsilon_i\cdot\epsilon_b)(p_i-p_b)^\mu + \epsilon_b^\mu(2p_b-p_i)\cdot\epsilon_i + \epsilon_i^\mu(-p_b-2p_i)\cdot\epsilon_b\Big) \tag{6}$$

Only the second term is non-vanishing.

So we can combine the divergent terms together.

$$\begin{aligned} -i\Big(&\frac{M(a,i)^\mu}{(p_i+p_a)^2} M(1,\dots,p_a+p_i,b,\dots n)_\mu + \frac{M(i,b)^\mu}{(p_i+p_b)^2} M(1,\dots,a,p_i+p_b,\dots n)_\mu\Big) \\ &\to \frac{1}{\sqrt{2}}\Big(-2(p_a\cdot\epsilon_i)\frac{A(1,\dots a,b,\dots,n)}{(p_a+p_i)^2} + 2(p_b\cdot\epsilon_i)\frac{A(1,\dots a,b,\dots,n)}{(p_i+p_b)^2}\Big) \end{aligned} \tag{7}$$

Then we need to specify the helicity of i.

$$\epsilon_i^+ = \sqrt{2}\frac{\lambda_k\tilde{\lambda}_i}{\langle ki\rangle}, \quad \epsilon_i^- = \sqrt{2}\frac{\lambda_i\tilde{\lambda}_k}{[ik]}, \tag{8}$$

It is convenient to choose $k=a$.

- i^- helicity. In this case, (7) becomes

$$A(1,\dots,a,i^-,b,\dots,n) \to -\frac{[ab]}{[ai][ib]} A(1,\dots,a,b,\dots,n)\,. \tag{9}$$

- i^+ helicity. In this case, (7) becomes

$$A(1,\dots,a,i^+,b,\dots,n) \to \frac{\langle ab\rangle}{\langle ai\rangle\langle ib\rangle} A(1,\dots,a,b,\dots,n)\,. \tag{10}$$

Note that the helicity for i is consistent in these formulae.

The amazing feature is that the soft limit formula only depends on the helicity of i, not on the helicities of a and b. These formulae are still true even if a and b are quarks. The reason is that *when the i-gluon has a long wavelength, the quantum feature is gone and the theory becomes classical.* In this situation, helicity as a quantum number, is no longer relevant.

In computation, we may need a rigorous kinematics for the soft limit. It is impossible to take $p_i \to 0$ without changing other momenta. Instead, we use the spinor table

$$
\begin{array}{c|c|c}
1 & 1 & \tilde{1} - |c|^2 z^2 \tilde{1} - c\bar{d} z^2 \tilde{2} \\
\hline
2 & 2 & \tilde{2} - \bar{c} d z^2 \tilde{1} - |d|^2 z^2 \tilde{2} \\
\hline
\ldots & \ldots & \\
\hline
i & z(c1 + d2) & z(\bar{c}\tilde{1} + \bar{d}\tilde{2}) \\
\hline
\ldots & \ldots &
\end{array}
\tag{11}
$$

Here all particles are on-shell and z is free parameter. The soft limit is rigorously defined as $z \to 0$.

II. COLLINEAR LIMIT

Similarly, we can consider the collinear limit when two massless particles become parallel (collinear).

For a tree gluon partial amplitude $A(1, \ldots i, i+1, \ldots, n)$. We take the limit when

$$p_i \to zP, \quad p_{i+1} \to (1-z)P \tag{12}$$

where $P^2 = 0$ is a null vector. That is, to take

$$
\begin{aligned}
\lambda_i \to z\lambda_P, &\quad \tilde{\lambda}_i \to z\tilde{\lambda}_P \\
\lambda_{i+1} \to (1-z)\lambda_P, &\quad \tilde{\lambda}_{i+1} \to (1-z)\tilde{\lambda}_P
\end{aligned}
\tag{13}
$$

In this limit

$$\epsilon_i^{\pm} \to \epsilon_P^{\pm}, \quad \epsilon_{i+1}^{\pm} \to \epsilon_P^{\pm} \tag{14}$$

Again, the amplitude $A(1, \ldots i, i+1, \ldots, n)$ becomes divergent in this limit. Only the p_i and p_{i+1} combined vertex contributes to the divergence. Hence, the collinear limit for tree amplitude reads,

$$A(1, \ldots i, i+1, \ldots, n) \to \sum_{h=\pm} \mathrm{Split}_h(i, i+1) A(1, \ldots, P^{-h}, \ldots, n) \tag{15}$$

$\mathrm{Split}_h(i, i+1)$ is called the tree-level splitting function.

It is possible to derive the splitting function from tree-level Feynman diagram analysis. This function depends on the helicity of both i and $i+1$. Here we just show one example, the determination of $\mathrm{Split}_-(i^+, (i+1)^+)$.

In the collinear limit, the tree amplitude factorizes as,

$$A(1,\dots i,i+1,\dots,n) \to M^{\mu}(i,i+1)\frac{-i}{2p_i\cdot p_{i+1}}M_{\mu}(1,\dots,p_i+p_{i+1},\dots,n). \tag{16}$$

From Feynman rules, we have,

$$M(i,i+1)^{\mu} = \frac{i}{\sqrt{2}}\Big((\epsilon_i\cdot\epsilon_{i+1})(p_i-p_{i+1})^{\mu}+\epsilon^{\mu}_{i+1}(2p_{i+1}+p_i)\cdot\epsilon_i+\epsilon^{\mu}_i(-2p_i-p_{i+1})\cdot\epsilon_{i+1}\Big). \tag{17}$$

Note that all terms are vanishing in the collinear limit. It does not mean that the divergence is gone. We needed a detailed analysis.

For the $(i^+,(i+1)^+)$ case, we choose the same reference vector p_k to remove terms proportional to $\epsilon_i\cdot\epsilon_{i+1}$. Then the second term is evaluated to

$$\begin{aligned}&\frac{1}{\sqrt{2}}\epsilon^{\mu}_{i+1}(2p_{i+1}\cdot\epsilon_i)\frac{1}{\langle i,i+1\rangle[i,i+1]}\\&=-\sqrt{2}\epsilon^{\mu}_{i+1}\frac{\langle i+1,k\rangle}{\langle k,i\rangle\langle i,i+1\rangle}\\&\to\epsilon^{+\mu}_{P}\frac{\sqrt{1-z}}{\sqrt{z}}\frac{1}{\langle i,i+1\rangle}\end{aligned} \tag{18}$$

The third term becomes

$$\epsilon^{+\mu}_{P}\frac{\sqrt{z}}{\sqrt{1-z}}\frac{1}{\langle i+1,i\rangle} \tag{19}$$

Combine them together, we have

$$\mathrm{Split}_{-}(i^+,(i+1)^+) = \frac{1}{\sqrt{z(1-z)}\langle i,i+1\rangle}\,, \tag{20}$$

$$\mathrm{Split}_{+}(i^+,(i+1)^+) = 0\,. \tag{21}$$

Note that the variable z appears in the splitting function. Like the soft limit, the helicities of other particles do not enter this function.

The divergence is coming from $1/\langle i,i+1\rangle$. This looks like a square root of $p_i\cdot p_j$. Therefore the collinear limit is not as "divergent" as the soft limit.

The derivation of other splitting functions are involved. Here we list the result [1],

$$\mathrm{Split}_{-}(i^-,(i+1)^-) = 0\,, \tag{22}$$

$$\mathrm{Split}_{+}(i^+,(i+1)^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\langle i,i+1\rangle}\,, \tag{23}$$

$$\mathrm{Split}_{-}(i^+,(i+1)^-) = -\frac{z^2}{\sqrt{z(1-z)}[i,i+1]}\,, \tag{24}$$

For example, we consider the collinear limit for $A(1^-2^-3^+4^+5^+)$ with,

$$\lambda_4 = z\lambda_P, \quad \tilde{\lambda}_4 = z\lambda_P \tag{25}$$

$$\lambda_5 = (1-z)\lambda_P, \quad \tilde{\lambda}_5 = (1-z)\tilde{\lambda}_P \tag{26}$$

The MHV amplitude factorize as,

$$\begin{aligned}
A(1^-2^-3^+4^+5^+) &= i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle \mathbf{34}\rangle\langle \mathbf{45}\rangle\langle \mathbf{5}1\rangle} && (27)\\
&= \frac{1}{\sqrt{z(1-z)}\langle 45\rangle} i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 3P\rangle\langle P1\rangle} && (28)\\
&= \frac{1}{\sqrt{z(1-z)}\langle 45\rangle} A(1^-2^-3^+P^+)\,. && (29)
\end{aligned}$$

Similarly, for

$$\lambda_5 = z\lambda_P, \quad \tilde{\lambda}_5 = z\lambda_P \tag{30}$$

$$\lambda_1 = (1-z)\lambda_P, \quad \tilde{\lambda}_1 = (1-z)\tilde{\lambda}_P \tag{31}$$

The MHV amplitude factorize as,

$$\begin{aligned}
A(1^-2^-3^+4^+5^+) &= i\frac{\langle \mathbf{1}2\rangle^4}{\langle \mathbf{1}2\rangle\langle 23\rangle\langle 34\rangle\langle 4\mathbf{5}\rangle\langle \mathbf{51}\rangle} && (32)\\
&= \frac{(1-z)^2}{\sqrt{z(1-z)}\langle 51\rangle} A(P^-2^-3^+4^+)\,. && (33)
\end{aligned}$$

These factorizations are consistent with the splitting functions.

The rigorous collinear kinematics can be for example parameterized as,

1	1	$\tilde{1} - \sqrt{z}t\tilde{\lambda}_P$
2	$2 - t\sqrt{1-z}\lambda_P$	$\tilde{2}$
$\ldots$	$\ldots$	
i	$\sqrt{z}\lambda_P + t1$	$\sqrt{z}\tilde{\lambda}_P$
$i+1$	$\sqrt{1-z}\lambda_P$	$\sqrt{1-z}\tilde{\lambda}_P + t\tilde{2}$
$\ldots$	$\ldots$	

(34)

where t is a free parameter for the collinear limit.

[1] L. J. Dixon (1996), hep-ph/9601359.